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The averaging bias - a standard miscalculation, which extensively underestimates real CO_2 emissions

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Summary

The substitution of energy based on fossil fuels in different sectors like household or traffic by electric energy saves CO_2 of this specific sector due to decreased fossil fuel consumption. An important quantity is the additional CO_2 emission $\Delta F(\bar{D}, \Delta D)$ due to an increased electric power demand ΔD for the average electricity power demand $\bar{D}$. Commonly, the formula $\Delta F(\bar{D}, \Delta D) \approx M(\bar{D})\Delta D$ is used (called simplified formula), where $M(\bar{D})$ represents mean average CO_2 footprint. It is shown in the present manuscript, that the simplified formula may underestimate the CO_2 footprint significantly if the average CO_2 footprint depends on the average electricity power demand, which is the case for most of mixed partly renewable and partly non-renewable electric energy systems. Therefore, the real CO_2 emissions would outmatch those according to simplified easily by factor 2 in reality depending on the status of the electricity system. In order to establish a more precise calculation of the CO_2 footprint, the general formula $\Delta F(\bar{D}, \Delta D) = \bar{D}\Delta M(\bar{D}, \Delta D) + \Delta D M(\bar{D} + \Delta D)$ which is exact and contains the simplified formula as a special case, is derived in this manuscript. The simplified formula requires an additional term that takes into account the change of the mean average CO_2 footprint ΔM depending on the electricity power demand.

KEYWORDS:
CO_2 emissions, electricity, fossil-based energy, non-fossil-based energy, fundamental theorem of differential and integral calculation according to Leibniz

1 | GENERAL INTRODUCTION

The rapid reduction of global CO_2 emissions is the key recommendation of the Intergovernmental Panel of Climate Change IPCC[1]. Policymakers around the world are responding to enable this ambitious target[2,3]. A total global remaining CO_2 budget of 420 Gt for all humanity was analyzed by the IPCC to limit global warming to 1.5 °C. Detailed probabilities for the achievement of the warming limit have been determined but are unimportant for the focus of this publication.

A policy approach to manage and analyze the reduction of CO_2 emissions is to define different sectors such as electric power, transport, industry, households. Each sector is typically regulated with a tighter limit on CO_2 emissions, i.e., a 50% reduction. However, looking at each sector in isolation can lead to inaccurate estimates of CO_2 emissions because the sectors interact.

An example is the heat supply of a building. The advisable substitution of an oil-burner by a modern heat-pump eliminates the CO_2 emissions of the sector "households", as the oil consumption is eliminated. As a consequence the power demand of the sector "electric energy" is increased in order to operate the new electric heat-pump. The CO_2 reduction of the sector "household" can be easily determined. For instance, a decrease of oil consumption of $\Delta V_{Oil}/\Delta t = -1000$ l/year leads to a decrease of CO_2 emissions of $\Delta m_{CO_2}/\Delta t = -3200\,\mathrm{kg/year}$.

But how does the additional demand for electrical energy increase CO_2 emissions from the "electrical energy" sector? For the sector "electric energy" a constant average CO_2 footprint M (unit: g_{CO_2}/kWh) is available. The standard calculation of the CO_2 impact of increased electric power demand ΔD (unit: GW) for a exemplary time period Δt is typically calculated as [4,5,6,7,8,9,10,11,12,13,14]:

$$\frac{\Delta m_{CO_2 sector_{electric\ energy}}}{\Delta t} = M\ \Delta D. \tag{1}$$

Please note that the unit of M respectively ΔD needs to be adapted in order to calculate the correct dimension of the result. The outline of the manuscript is as follows: Section 2 gives an overview with respect to the different energy sectors and the corresponding CO_2 footprint. In section 3, the fundamental theorem of calculus is used to relate the CO_2 impact due to an increased electric power demand to the average CO_2 footprint M. It is shown that the dependence of the average CO_2 on the power demand ΔD has to be taken into account in order not to underestimate the CO_2 footprint. Several examples are discussed. Section 4 summarizes the results.

2 | ANALYSIS

In order to derive the CO_2 emissions of the sector "electric energy", the characteristics of electric power generation must be analyzed. An hourly resolved matrix P_{ij} electric power generation for Germany in the year 2017 is the basis of the analysis [15] with i specifying the electriciy source and j the hour in one representative year. For $j \in [1, 8760]$ hours, the electric power P_{ij} of 8 electricity sources i is known. The year 2017 has been chosen, as the Matrix P_{ij} of 2017 has been the latest available hourly resolved complete dataset. However, the chosen year does not influence the general analysis of the averaging bias at all. The CO_2 impact for all technologies i is also depicted in Table 1.

As an example; P_{1j} denotes Wind Power, P_{8j} denotes Brown Coal Power for each hour j. "Regenerative Power" P_j^{reg}, "Non Regenerative Power" P_j^{nreg} and "Supply" S_j are defined as hourly averaged values as follows

$$P_j^{\mathrm{reg}} = \sum_{i=1}^{4} P_{ij}, \tag{2}$$

$$P_j^{\mathrm{nreg}} = \sum_{i=5}^{8} P_{ij}, \tag{3}$$

$$S_j = \sum_{i=1}^{8} P_{ij} = \sum_{i=1}^{4} P_{ij} + \sum_{i=5}^{8} P_{ij} = P_j^{\mathrm{reg}} + P_j^{\mathrm{nreg}}. \tag{4}$$

Note that the expression "regenerative energy" is not correct from the thermodynamic perspective. Nevertheless it is used in this manuscript as it represents a common definition for photovoltaics, wind, water and biomass based electric power.

The indices depict an average value over one hour, i.e. $P_{3\ 1849}$, represents the average "Photovoltaics Power" for hour 1849. Please note, that the consideration of import and export energy transport via the system boundary requires an extra balance factor B_j. Also an additional energy storage capacity based power storage and supply contribution P_j^{st} will be necessary. Losses due to electric resistance and the electric power transformation are denoted by H_j. Therefore, the electric energy demand D of hour j is defined as a function of the relevant energy contributors as

$$D_j = P_j^{\mathrm{reg}} + P_j^{\mathrm{nreg}} + B_j + P_j^{\mathrm{st}} + H_j = S_j + B_j + P_j^{\mathrm{st}} + H_j. \tag{5}$$

The yearly average of B_j amounts to roughly 10% of the yearly average total energy demand D_j. H_j typically scales between 4 to 10% of D_j [11,16]. Within the next decades the energy storage capacity based power request and support P_j^{st} becomes more important as the following situation will occur more and more frequently, especially after the year 2030

$$P_j^{\mathrm{reg}} > D_j. \tag{6}$$

However, for the derivation of the averaging bias, the import/export balance B_j, the electric resistance H_j as well as the energy storage P_j^{st} are set equal to zero, which implies $S_j = D_j$. In general, the supply is a function of the energy demand, since the supply must satisfy the demand. Following equation (5), the supply of energy S_j is a function of energy demand D_j, with priority of energy contributor P_j^{reg}. Index j again denotes the average value of a certain hour j. This can be expressed as

$$S_j = f_1(D_j), \tag{7}$$

$$P_j^{\mathrm{reg}} = f_2(\text{weather, status electricity grid, ... }), \tag{8}$$

$$P_j^{nreg} = f_3(D_j, P_j^{reg}). \tag{9}$$

The functions f_1, f_2, f_3 symbolize the general and complex dependency of electric demand and supply as well as the interaction between weather and boundary conditions on P_j^{reg} and the resulting dependency of P_j^{nreg}.

TABLE 1 CO_2 equivalents CO_{2i} of different technologies, according to[17]; oil is of minor importance and neglected in this publication. (* Wind Power technology includes the respective contributions of onshore and offshore.)

i	**Technology**	$\bar{E}_i^{CO_2}$ in g_{CO_2} /kWh	**Classification**
1	Wind Power *	9	regenerative
2	Hydropower	23	regenerative
3	Photovoltaics	50	regenerative
4	Biomass	70	regenerative
5	Nuclear	24	non regenerative
6	Gas	499	non regenerative
7	Hard Coal	830	non regenerative
8	Brown Coal	1075	non regenerative

Fig. 1 depicts the contribution of different energy sources to power generation in the year 2017. The transient behavior of P_{1j} (Wind), P_{8j} (Brown Coal) as well as P_j^{reg} (Sum Regenerative) and P_j^{nreg} (Sum Non Regenerative) can be seen. Already in the year 2017 an impressive contribution of renewable electric energy P_j^{reg} was established. Of course a rising importance of P_j^{reg} is anticipated according to[18,19].

The electricity demand D is not plotted in Fig. 1 , but fluctuates between 40 and 80 GW within a year. The renewable energy supply P_j^{reg} varies between 7.7 and 61 GW, wherein biomass enables a regenerative baseload. Non regenerative energy is typically needed to close the gap with a P_j^{nreg} peak of 62 GW. The following analysis considers an unlimited energy transport within the system boundaries.

In order to determine the CO_2 impact of the complete sector “electric energy”, the detailed contribution of different electric energy sources must be considered as defined in Table 1. The combination of a detailed knowledge of each electric energy source (i.e. Hydropower, Wind, Gas, …) and the specific CO_2 equivalent impact of each technology CO_{2i} enables the calculation of CO_2 emissions of P_j^{reg}, P_j^{nreg} and S_j, according to equations (10)-(12) for every hour j. The result is depicted in Fig. 2 . The necessary equations are

$$E_j^{\mathrm{reg}\,\mathrm{CO_2}} = \frac{1}{\sum_{i=1}^{4} P_{ij}} \sum_{i=1}^{4} P_{ij} \cdot \bar{E}_i^{\mathrm{CO_2}} = \frac{1}{P_j^{\mathrm{reg}}} \sum_{i=1}^{4} P_{ij} \cdot \bar{E}_i^{\mathrm{CO_2}}, \tag{10}$$

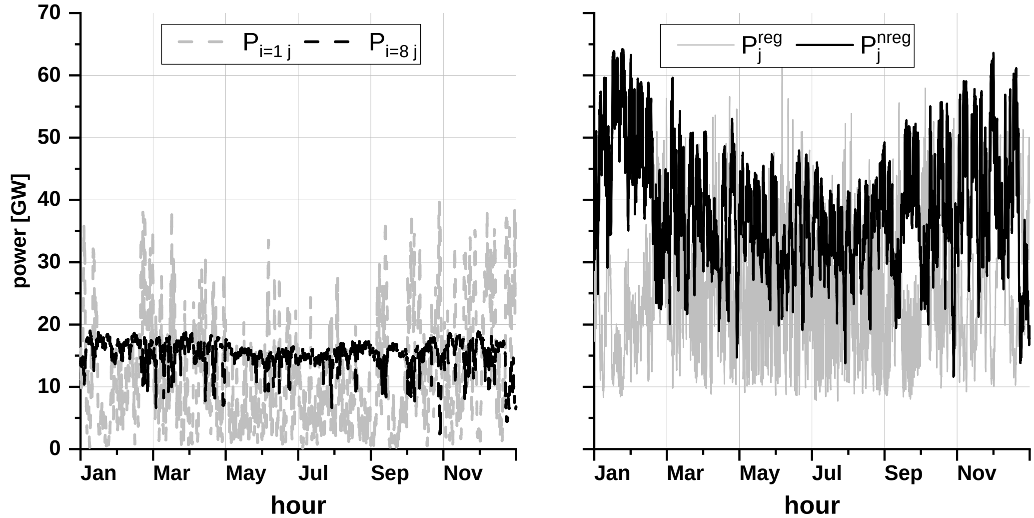

FIGURE 1 Evolution of electric power generation over 8760 hours; Evolution of P_{1j} (Wind), P_{8j} (Brown Coal), P_j^{reg} (Sum Regenerative) and P_j^{nreg} (Sum Non Regenerative); Source: Germany 2017 data according to[18,19].

$$E_j^{\mathrm{nreg\,CO_2}} = \frac{1}{\sum_{i=5}^{8} P_{ij}} \sum_{i=5}^{8} P_{ij} \cdot \bar{E}_i^{\mathrm{CO_2}} = \frac{1}{P_j^{\mathrm{nreg}}} \sum_{i=5}^{8} P_{ij} \cdot \bar{E}_i^{\mathrm{CO_2}} \quad \text{and} \tag{11}$$

$$E_j^{\mathrm{tot\,CO_2}} = \frac{1}{\sum_{i=1}^{8} P_{ij}} \sum_{i=1}^{8} P_{ij} \cdot \bar{E}_i^{\mathrm{CO_2}}. \tag{12}$$

Note that P_j^{reg} is mainly depending on the weather and the status of the electric grid. But especially P_j^{nreg} is a function of energy demand D_j. Therefor $E_j^{\mathrm{nreg\,CO_2}}$ as well as $E_j^{\mathrm{tot\,CO_2}}$ are also a function of energy demand D_j.

In order to define the CO_2 emissions as a function of the hourly averaged electric power demand D_j, two different possibilities are presented in equation (13)-(15). In the first approach one assumes

$$E_j^{\mathrm{CO_2}} = \bar{E}_{k_j^{\mathrm{min}}}^{\mathrm{CO_2}} \tag{13}$$

with $1 \le k_j^{\mathrm{min}} \le 8$ defined for each hour j by as the minimum k satisfying

$$\sum_{i=1}^{k} P_{ij} \ge D_j. \tag{14}$$

Equation (13) and the condition (14) imply, that regenerative energy has priority and within the electricity system and for a given electricity demand D_j the electricity contribution of technologies i with lowest CO_2 impact is supplied with priority.

Fig. 3 illustrates, that for hour 4899 and 3780 the theoretical behavior of the CO_2 impact $E_j^{\mathrm{CO_2}}$ is illustrated. As the renewable energy is typically not sufficient as $D_j > P_j^{reg}$, additional energy P_j^{reg} must be provided. Therefore, a decrease of CO_2 impact is depicted because of nuclear power ($i = 5$) while afterwards the CO_2 impact increases again up to a specific brown coal energy value of 1075 g_{CO_2}/kWh.

Indeed equation (13) is only the consequence of the aforementioned theoretical assumption, that only the technology i with the lowest CO_2 impact is applied step by step. More technologies i would be added consecutively to satisfy the demand in theory. However, most electricity contributors co-contribute simultaneously due to electricity net constraints and long distance electricity transport challenges. Therefore the CO_2 impacts of $E_j^{\mathrm{reg\,CO_2}}$ and $E_j^{\mathrm{nreg\,CO_2}}$ are determined in a second approach as a

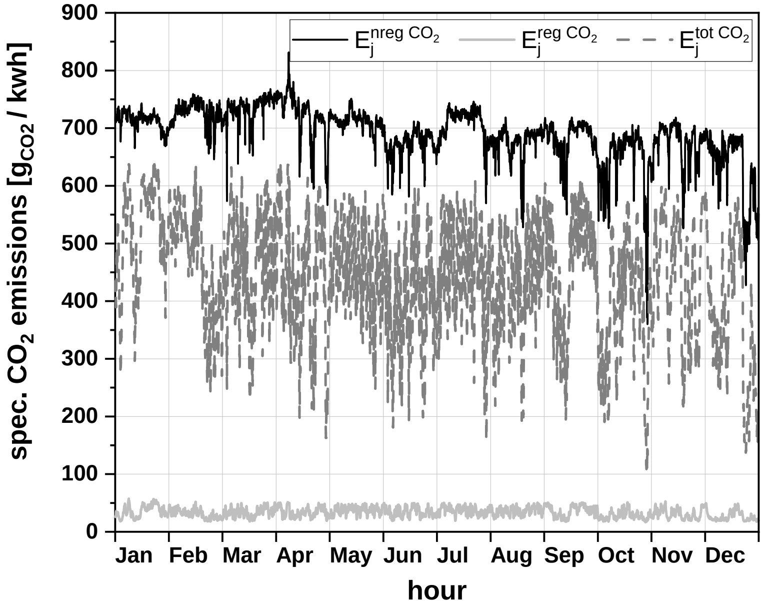

FIGURE 2 CO_2 emissions of electric power production of 8760 hours, evolution of E_j^{reg}, E_j^{nreg} and E_j^{tot}, year 2017.

function of electric power generation P_j, which is equivalent to the power demand D for the given assumptions

$$E_j^{\text{CO}_2\text{ cluster}}(D_j) = \begin{cases} E_j^{\text{reg CO}_2}, & D_j \leq P_j^{\text{reg}}, \\ E_j^{\text{nreg CO}_2}, & D_j > P_j^{\text{reg}}. \end{cases} \tag{15}$$

Equation (15) represents a further adaption to realistic boundary conditions as all regenerative contributors are summarizes for low electric power demand and all non regenerative contributors are bundled for high electric power demand. The result of equation (13)-(15) is depicted in Fig. 3 . It illustrates selected representative elements of the 8760 hour matrix.

The results for both, equation (13) and (15), respectively, are shown for hour 4899 and 3780. The minimal regenerative power ($j = 4899$) of 7.7 GW at night with a contribution of hydropower ($i = 2$) and biomass ($i = 4$) is quite limited. The result $E_j^{\text{CO}_2\text{ Cluster}}$ of the alternative equation 15 is plotted additionally and depicts the low regenerative CO_2 footprint and a non regenerative average specific emission of 741 g_{CO_2}/kWh. Both results are also plotted for hour $j = 3780$ with a maximal regenerative power of 61.3 GW, wherein wind ($i = 1$) and photovoltaics ($i = 3$) dominate the regenerative contribution. Note that the non regenerative footprint of 604 g_{CO_2}/kWh is smaller compared to hour $j = 4899$. The relative contribution of nuclear power ($i = 5$) to the total non regenerative power supply in hour $j = 3780$ causes the difference. Hour $j = 2251$ represents the highest total energy specific footprint $E_j^{\text{tot CO}_2}$ of 644 g_{CO_2}/kWh as the regenerative output contributes only 9.8 GW but the total demand was 60.1 GW. On the other hand side hour $j = 7236$ illustrates the lowest total energy specific footprint $E_j^{\text{tot CO}_2}$ of 96.7 g_{CO_2}/kWh with a dominant regenerative contribution of 53.1 GW. Finally, hour 2361 illustrates the highest non regenerative specific footprint $E_j^{\text{nreg CO}_2}$ of 831.5 g_{CO_2}/kWh due to a dominant coal energy contribution.

The yearly averaged CO_2 impact $E_{2017}^{CO_2}$ as a function of the energy demand D can be calculated by the following equation

$$E_{2017}^{CO_2}(D) = \frac{1}{8760} \sum_{j=1}^{8760} E_j^{CO_2\ cluster}(D_j). \tag{16}$$

Note that D is identical to the total electric power generation P according to the simplifying assumptions and equation (5).

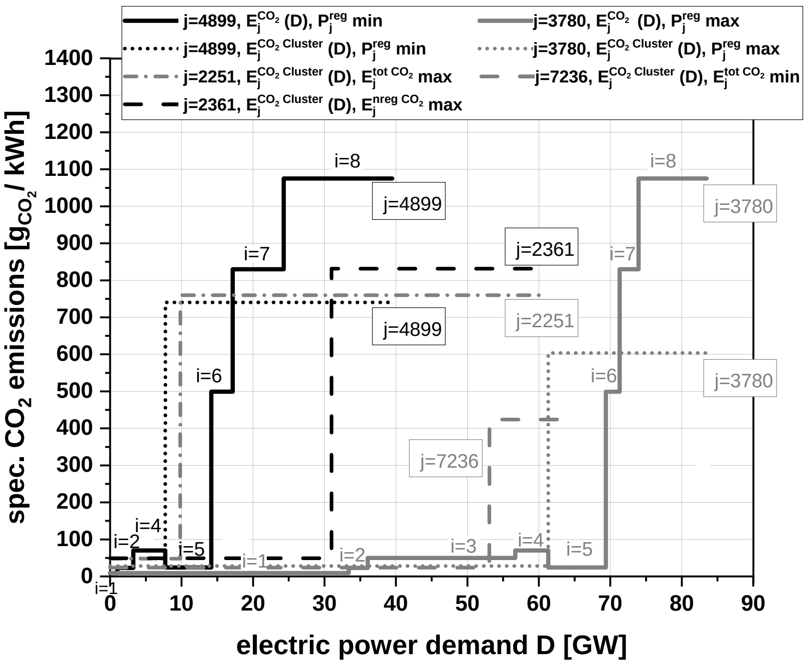

FIGURE 3 Year 2017: CO_2 emissions of selected hours as a function of electric energy demand
$j = 4899$, 24.7.2017; 03.00-04.00: minimal regenerative power P_j^{reg}; $P_j^{reg} = 7.7$ GW, note: $P_3 = 0$
$j = 3780$, 7.6.2017; 12.00-13.00: maximal regenerative power P_j^{reg}; $P_j^{reg} = 61.3$ GW
$j = 2251$, 4.4.2017; 20.00-21.00: worst total energy mix $E_j^{tot\,CO_2} max$; $E_j^{tot\,CO_2} = 643.6$ g_{CO_2}/kWh
$j = 7236$, 29.10.2017; 13.00-14.00: best total energy mix $E_j^{tot\,CO_2} min$; $E_j^{tot} = 96.7$ g_{CO_2}/kWh
$j = 2361$, 9.4.2017; 09.00-10.00: worst non regenerative energy mix $E_j^{nreg} max$; $E_j^{nreg} = 831.5$ g_{CO_2}/kWh.

Furthermore, the moving average value $E_{2017}^{Av\,CO_2}$ is defined as

$$E_{2017}^{Av\,CO_2}(\bar{D}) = \frac{1}{\bar{D}} \int_0^{\bar{D}} E_{2017}^{CO_2}(\widetilde{D})\ \mathrm{d}\widetilde{D} \tag{17}$$

with the yearly averaged electricity demand $\bar{D}$

$$\bar{D} = \frac{1}{8760} \sum_{j=1}^{8760} D_j. \tag{18}$$

Please note that $E^{CO_2}(\widetilde{D})$ depends on the year via weather conditions, technology change and the adapted demand, indicated by an index denoting the year, e.g., by $E_{2017}^{CO_2}(\widetilde{D})$ and later by $E_{2030}^{CO_2}(\widetilde{D})$. Note that $E_{2017}^{Av\,CO_2}(\bar{D})$ corresponds to M in equation (1).
Fig. 4 illustrates the results of equation (16) and (17) for the data of 2017. The evolution $E_j^{\mathrm{CO_2\ cluster}}$ of selected hours were discussed in Fig. 3 but are replotted on the left hand side of Fig. 4 in order to demonstrate the derivation of $E_{2017}^{\mathrm{CO_2}}$. Also the moving average value $E_{2017}^{\mathrm{Av\,CO_2}}$ is plotted according to equation (17) on the right hand side.
In addition a simulation of the year 2030 has been accomplished[20] with detailed information about the scale up of regenerative power installation according to[18,19]. Furthermore, the increase of energy storage capacities is considered as well as the increase of electricity demand due to ambitious heat pump or battery electric vehicle penetration scenarios with increased $\bar{D}$ as a consequence. These results are also depicted in Fig. 4 . However detailed explanations of the 2030 calculation are not in the focus of this publication, as the general analysis is of major interest.

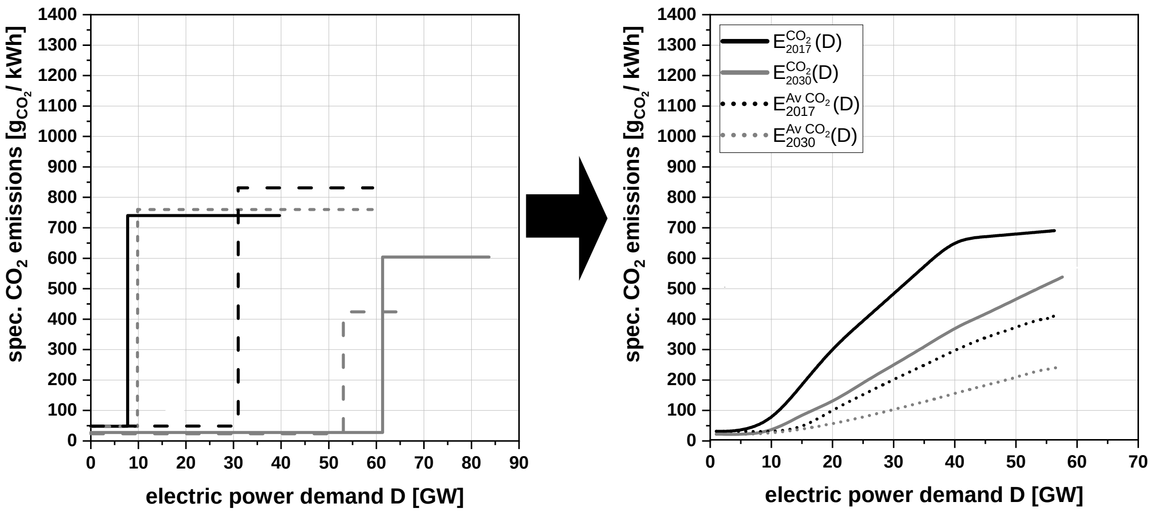

FIGURE 4 left: examples of $E_j^{\mathrm{CO_2 cluster}}$ of the year 2017 as in Fig. 3 ;
right: evolution of $E_{2017}^{\mathrm{CO_2}}$ and $E_{2017}^{\mathrm{Av\,CO_2}}$, 2017: $\bar{D}$=56.3 GW, $E_{2017}^{\mathrm{Av\,CO_2}}$ (56.3 GW)=412 g_{CO_2}/kWh;
evolution of $E_{2030}^{\mathrm{CO_2}}$ and $E_{2030}^{\mathrm{Av\,CO_2}}$, 2030: $\bar{D}$ = 57.6 GW; $E_{2030}^{\mathrm{Av\,CO_2}}$(57.6 GW) = 244 g_{CO_2}/kWh.

The main question remains the analysis of the CO_2 impact of an increased electricity demand ΔD. Substituting $\Delta m_{CO_2 sector_{electric\ energy}}$ in the following by the simplified notation Δm_{CO_2} leads to the commonly used equation (see also equation (1)): $\Delta m_{CO_2} = M\,\Delta D\,\Delta t$. Indeed the correct calculus i.e. for the year 2030 is defined as shown in equation (19).

$$\Delta m_{CO_2} = \Delta t \int_{\bar{D}}^{\bar{D}+\Delta D} E_{2030}^{CO_2}(\tilde{D})\ \mathrm{d}\tilde{D} \tag{19}$$

Besides $E_{2017}^{CO_2}(D)$ and $E_{2017}^{Av\,CO_2}(D)$ the derivative of $\mathrm{d}E_{2017}^{Av\,CO_2}(D)/\,\mathrm{d}D$ becomes of major importance, which is explained in the next section. Note that $\mathrm{d}E_{2017}^{\mathrm{Av\,CO_2}}(D)/\,\mathrm{d}D$ is equivalent to the derivative $\mathrm{d}M(D)/\,\mathrm{d}D$ according to the nomenclature of equation (1), which is equivalent to $\mathrm{d}M(x)/\,\mathrm{d}x$ of the general formula (see, (31) in section 3.) Also note that the importance of this derivative remains important over the years, especially for an electric power demand in the range of 60GW. Although $E_{2030}^{CO_2}(D)$ and $E_{2030}^{Av\,CO_2}(D)$ are significantly smaller than in the year 2017, the derivative $\mathrm{d}E_{2030}^{Av\,CO_2}(D)/\,\mathrm{d}D$ becomes more important, as $E_{2030}^{Av\,CO_2}(D)$ shows even a slightly steeper gradient in the relevant range of D of 60 GW, which is depicted in Fig. 5 .

3 | MATHEMATICAL FORMULATION OF THE PROBLEM

3.1 | Fundamental theorem of calculus

The fundamental theorem of calculus (Erster Hauptsatz der Differential- und Integralrechnung) relates the two fundamental concepts of calculus, that of integration and that of differentiation. It states that derivation and integration are mutual inverses (up to a constant)[21,22]. The theorem is stated in the following: Let $I = [a, b]$ be a closed interval on the real line $\mathbb{R}$ and $c \in I$.

TABLE 2 Definition of variables.

identifier 1	identifier 2	variable	unit	dimension
D	x, s	electric energy demand	W	$\mathrm{M} \cdot \mathrm{L}^2 \cdot /\mathrm{T}^3$
$E^{CO_2}_{2017}$, $E^{CO_2}_{2030}$	$f(x)$, $f(s)$	yearly averaged CO_2 impact as function of the energy demand D	g_{CO_2}/kWh	$\mathrm{M}_{\mathrm{CO_2}}$ /(M · L^2/T^2)
$\Delta m_{CO_2}/\Delta t$	$F(x)$	CO_2 impact per time period	g_{CO_2}/h	$\mathrm{M}_{\mathrm{CO_2}}$ /T
$E^{Av\,CO_2}_{2017}$	$M(x) = M(0, x)$	average value of CO_2 impact as function of energy demand D	g_{CO_2}/kWh	$\mathrm{M}_{\mathrm{CO2}}$ /(M · L^2/T^2)

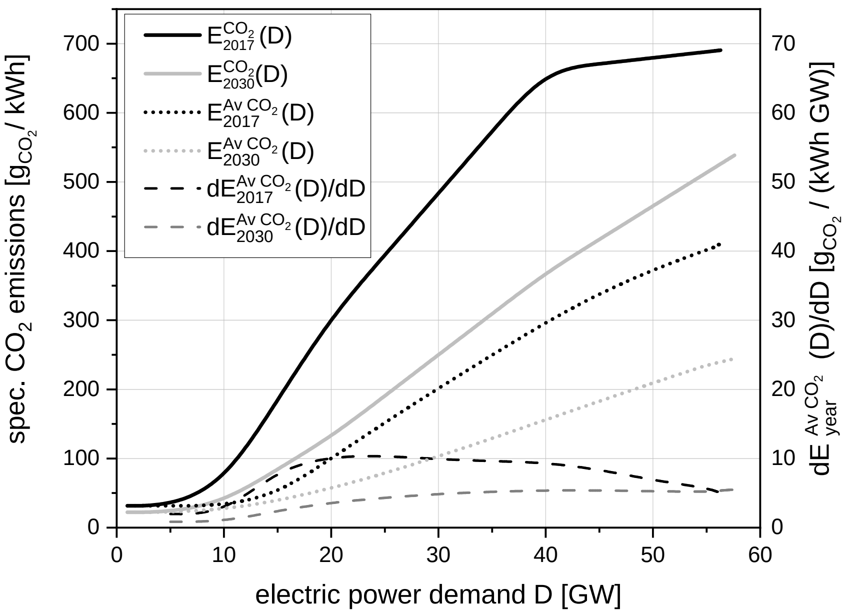

FIGURE 5 $E^{\mathrm{CO_2}}_{2017}(D)$, $E^{\mathrm{CO_2}}_{2030}(D)$ and $E^{\mathrm{Av\,CO_2}}_{2017}(D)$, $E^{\mathrm{Av\,CO_2}}_{2030}(D)$ as in Fig. 4 (right diagram), $\mathrm{d}E^{\mathrm{Av\,CO_2}}_{2017}(D)/\,\mathrm{d}D$ and $\mathrm{d}E^{\mathrm{Av\,CO_2}}_{2030}(D)/\,\mathrm{d}D$ on the right axis. Please note that $\mathrm{d}E^{\mathrm{Av\,CO_2}}_{j}(D)/\,\mathrm{d}D$ is equivalent to the simplified derivative $\mathrm{d}M/\,\mathrm{d}D$ of equation 1 respectively $\mathrm{d}M/\,\mathrm{d}x$ of the general formula (see, equation (31)).

Furthermore, let $f : I \to \mathbb{R}$ be a real-valued piecewise continuous function defined on I. Then, the function

$$F(x) = \int_c^x f(s)\,\mathrm{d}s \tag{20}$$

is continuous on I and continuously differentiable on the open interval (a, b) with

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = F'(x) = f(x). \tag{21}$$

The total differential of $F(x)$ is

$$\mathrm{d}F(x) = F'\,\mathrm{d}x = f(x)\,\mathrm{d}x. \tag{22}$$

If $f(x)$ can be assumed to be nonnegative, then the integral $F(x)$ can be interpreted based on the area under the curve $f(x)$ in the interval $I = [c, x]$. This implies that an increment $\mathrm{d}F(x) = F'(x)\,\mathrm{d}x$ represents an infinitesimal small area in the range x and $x + \mathrm{d}x$.

Example 3.1a: One may consider the linear function $f(x) = \alpha x$ with the real constant $\alpha > 0$ and the interval $I = [c, x]$. Then

$$F(x) = \frac{\alpha}{2}\left(x^2 - c^2\right) \tag{23}$$

holds. The derivative of $F(x)$ is

$$F'(x) = \alpha x, \tag{24}$$

and the total differential of $F(x)$ takes the form

$$\mathrm{d}F(x) = \alpha x\,\mathrm{d}x. \tag{25}$$

Example 3.1b: In the context of estimating the specific CO_2 footprint due to a power demand, one may specifically consider x as electrical power demand D in kW, $f(x)$ as the specific CO_2 emission in g_{CO_2}/kWh and $F(x)$ as the CO_2 footprint (CO_2 impact) within the range of energy demand $c = D_1$, $x = D_2$ with $D_1 \leq D_2$.

3.2 | Implication of the fundamental theorem for moving averages

The mean value M of the function $f(x)$ on the interval $I = [a, b]$ computes as

$$M(a, b) = \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s. \tag{26}$$

For the special case that $a = 0$ and $b = x$, one obtains for variable x a moving average

$$M(0, x) = \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s, \tag{27}$$

or, equivalently, with $M(x) = M(0, x)$,

$$x\ M(x) = \int_0^x f(s)\,\mathrm{d}s. \tag{28}$$

Determining the total differential of both sides of (28) gives, taking into account the relations (21) and (22) (note that $\mathrm{d}F(x) = f(x)\,\mathrm{d}x$),

$$\mathrm{d}(xM(x)) = f(x)\,\mathrm{d}x = \mathrm{d}F(x). \tag{29}$$

Applying the product rule

$$\mathrm{d}(xM(x)) = M(x)\,\mathrm{d}x + x\,\mathrm{d}M(x) \tag{30}$$

gives (*general formula*)

$$M(x)\,\mathrm{d}x + x\,\mathrm{d}M(x) = f(x)\,\mathrm{d}x = \mathrm{d}F(x), \tag{31}$$

which states that the increment of $F(x)$ is equal to the sum of the increment of x multiplied by M and the increment of M mulitplied by x. The special case (*simplified formula*)

$$M(x)\,\mathrm{d}x = f(x)\,\mathrm{d}x = \mathrm{d}F(x) \tag{32}$$

is only valid, if $|\,\mathrm{d}M(x)| \ll 1$ holds exactly or approximately.

Equation (31) can be reformulated using finite increments. With

$$F(x) = \int_0^x f(s)\,\mathrm{d}s, \qquad \Delta F(x, \Delta x) = \int_x^{x+\Delta x} f(s)\,\mathrm{d}s \tag{33}$$

one obtains

$$M(x) = \frac{F(x)}{x}, \qquad M(x + \Delta x) = \frac{F(x) + \Delta F(x, \Delta x)}{x + \Delta x} \tag{34}$$

or, equivalently,

$$xM(x) = F(x), \qquad (x + \Delta x)M(x + \Delta x) = F(x) + \Delta F(x, \Delta x). \tag{35}$$

Taking the difference of the last two equations results in

$$\boxed{x\Delta M(x,\Delta x) + \Delta x M(x+\Delta x) = \Delta F(x,\Delta x)} \tag{36}$$

with

$$\Delta M(x,\Delta x) = M(x+\Delta x) - M(x). \tag{37}$$

In simplified notation equation (36) may be recast as

$$\boxed{x\Delta M + \Delta x M = \Delta F,} \tag{38}$$

which will be called *general formula for estimating the CO_2 impact* in the following. The equation (38) is valid for increments of arbitrary size, but the arguments of the functions entering in equation (36) have to be taken into account carefully.

It should be noted that, in the context of estimating the CO_2 footprint, the *simplified formula*

$$\boxed{\Delta x M \approx \Delta F} \tag{39}$$

is commonly used. In particular, by (39), the increase in $\Delta F(x,\Delta x)$ may be severely underestimated. As demonstrated in the previous section, such positive values are not uncommon.

Example 3.2a: Assume the function $f(x)$ to be constant on the interval $I = [0,\infty)$: $f(x) = f_0$. Then, the mean value of the function is constant ($\mathrm{d}M(x) = 0$) and equal to the constant value of the function: $M(x) = f_0 = M_0$. As a result, the equations

$$M(x)\,\mathrm{d}x = f(x)\,\mathrm{d}x \tag{40}$$

and

$$\Delta x M = \Delta F \tag{41}$$

hold exactly. Therefore, for the special case of constant functions $f(x)$, the simplified formula (39) for the CO_2 impact is exact.

Example 3.2b: Assume that the function $f(x)$ is piecewise constant except for a jump at $x = x_0$ from 0 to $f_0 > 0$, i.e.,

$$f(x) = \begin{cases} 0, & x \le x_0, \\ f_0, & x > x_0. \end{cases} \tag{42}$$

Then it follows for $F(x)$ and for the moving average $M(x)$

$$F(x) = \begin{cases} 0, & x \le x_0, \\ f_0(x - x_0), & x > x_0 \end{cases} \tag{43}$$

and

$$M(x) = \begin{cases} 0, & x \le x_0, \\ f_0\frac{x-x_0}{x}, & x > x_0. \end{cases} \tag{44}$$

At $x = x_0$ the moving average changes from zero to positive values with slope $M'(x = x_0) = f_0/x_0$. This implies that, for a large jump f_0, there is a rapid change of the mean value close to $x = x_0$. Additionally, the approximation $\Delta F(x) \approx M(x)\Delta x$ is clearly inaccurate.

Example 3.2c: Again consider the linear function $f(x) = \alpha x$ or $\mathrm{d}f(x) = \alpha\,\mathrm{d}x$ involving a positive constant $\alpha > 0$. This implies $M(x) = M(0,x) = \alpha x/2$ and $\mathrm{d}M(x) = \alpha\,\mathrm{d}x/2$.

It follows that the general formula (see equations (30), (31), (38))

$$\mathrm{d}(xM(x)) = f(x)\,\mathrm{d}x \tag{45}$$

is naturally satisfied, whereas the simplified formula (see equations (32), (39))

$$M(x)\,\mathrm{d}x = f(x)\,\mathrm{d}x \tag{46}$$

is not fulfilled. Indeed, for general α, the term $x\,\mathrm{d}M(x)$, i.e., the change of $M(x)$ with x, is not taken into account.

For the integral one obtains

$$\Delta F(x,\Delta x) = \int_{x_0}^{x_0+\Delta x} \alpha x\,\mathrm{d}x = \alpha x_0 \Delta x + \frac{\alpha \Delta x^2}{2} = f_0\Delta x + \frac{\Delta f}{2}\Delta x. \tag{47}$$

Based on $f_0 = \alpha x_0$, $\Delta f = \alpha \Delta x$ and $M(x_0) = f_0/2$, this result may be decomposed into

$$\Delta F(x, \Delta x) = M(x_0)\Delta x + \frac{f_0}{2}\Delta x + \frac{\Delta f}{2}\Delta x. \tag{48}$$

With this example in mind it becomes clear, that the average $M(x_0)$ multiplied by Δx as an estimator for $\Delta F(x, \Delta x)$, i.e., $\Delta F(x, \Delta x) \approx M(x_0)\Delta x$, produces an erroneous result, because the terms $f_0\Delta x/2$ and $\Delta f \Delta x/2$ have been neglected.

4 | DISCUSSION AND CONCLUSION

For the calculation of CO_2 emissions of additional electric energy demand, insufficient simplified mathematic models are typically used, which might be motivated by the complexity of the electricity supply sources and the grid situation. An example for such a simplified fomula to analyze the additional CO_2 emissions per time interval $\Delta F(\bar{D},\Delta D)$ caused by additional electric power ΔD (unit: Watt) is the direct utilization of the average CO_2 emission footprint $M(\bar{D})$ (unit g_{CO_2}/kWh) for a given average electricity demand $\bar{D}$ of the electricity sector by the equation

$$\Delta F(\bar{D}, \Delta D) \approx M(\bar{D})\Delta D, \tag{49}$$

which corresponds to the simplified formula introduced in section 3, (see equation (39)). As shown in section 3, the following integral would be the exact formulation

$$\Delta F(\bar{D}, \Delta D) = \int_{\bar{D}}^{\bar{D}+\Delta D} f(D)\,\mathrm{d}D. \tag{50}$$

Here, $f(D)$ represents the specific CO_2 emissions as a function of electric power demand D.

The mathematical analysis showed, that equation (49) is only valid, when the CO_2 emissions are completely independent from the energy supply situation, i.e., if the complete electric energy would be either supplied constantly only by one technology, i.e., wind power, or would be supplied by a constant mix of several technologies, i.e. a combination of wind power and photovoltaics power, which is both by far not the case.

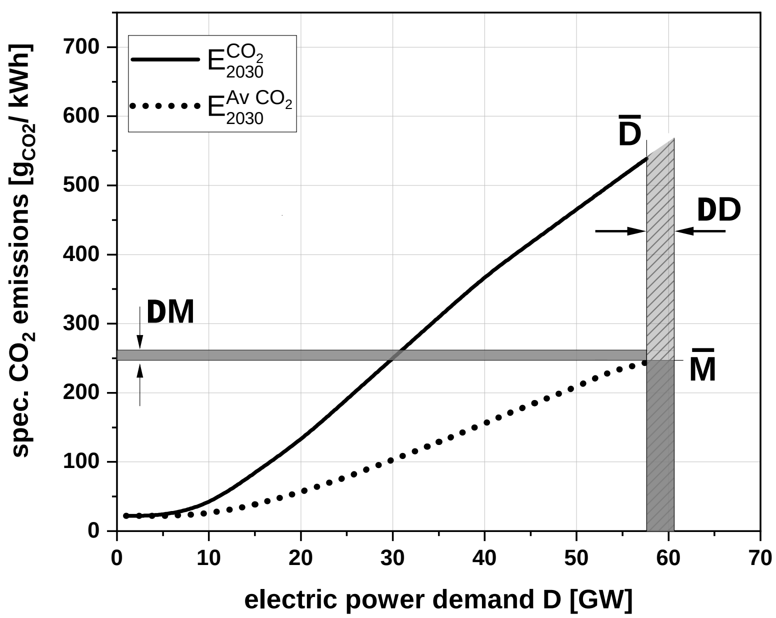

FIGURE 6 Graphical illustration of equation (50) and (51) Please note that the depicted areas represent $\bar{M}(\bar{D})\Delta\bar{D}$ and $\bar{D}\Delta M(\bar{D}, \Delta\bar{D})$.

The examples discussed in section 3 show for the specific assumption of a discontinuous, piecewise constant function and a linear function that the simplified formula is generally invalid and leads to erroneous results. The simplified formula is only valid for a constant function. Indeed, there is a clear interaction between electric power demand D and CO_2 emissions of the electricity sector, as additional electric energy supply typically requires the support of additional fossil power plants also in the future. It is clear that equation (49) cannot be generally utilized as it may significantly underestimates real CO_2 emissions.

By applying the fundamental theorem of differential and integral calculation of Leibniz of the 17^{th} century, the general and exact formula can be written as follows (see equations (36), (38))

$$\Delta F(\bar{D}, \Delta D) = \bar{D}\Delta M(\bar{D}, \Delta D) + \Delta D M(\bar{D} + \Delta D). \tag{51}$$

The term $\Delta M(\bar{D}, \Delta D)\bar{D}$ is missing in the simplified formula (49) and is important for most of mixed partly renewable and partly non-renewable electric energy systems. It can be even significantly larger than the term $\Delta D M(\bar{D} + \Delta D)$. Fig. 6 illustrates the contribution of both terms in order to define the increase of CO_2 emissions, according to equations (50) and (51). Note that the light grey area is equivalent to the left grey area, which represents the summand $\bar{D}\Delta M$ of equation (51) and the error of the simplified equation (49) The real CO_2 emissions of the electricity system may be significantly underestimated if only the simplified formula (49) is utilized. The real CO_2 emission would outmatch those according to the simplified equation (49) easily by factor 2 in reality depending on the status of the electricity system.

5 | NOTATION

Table 3 in addition to Table 2 explains the definition of major variables.

TABLE 3 Notation

symbol	physical quantity	unit	dimension
M	average CO_2 footprint	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
$\Delta m_{CO_2 sector_{electric\ energy}}$	CO_2 impact of the sector electric energy	g_{CO_2}	M_{CO_2}
B_j	extra balance power considering i.e. import / export	GW	$M \cdot L^2 \cdot T^{-3}$
$\bar{E}_i^{CO_2}$	average specific CO_2 equivalent impact of each technology i	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
D_j	electric energy demand	GW	$M \cdot L^2 \cdot T^{-3}$
$\bar{D}$	yearly averaged electricity demand	GW	$M \cdot L^2 \cdot T^{-3}$
$E_i^{reg\ CO_2}$	specific CO_2 impact of regenerative electric power	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
$E_i^{nreg\ CO_2}$	specific CO_2 impact of non regenerative electric power	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
$E_i^{tot\ CO_2}$	specific CO_2 impact of total electric power	g_{CO_2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
$E_j^{CO_2\ cluster}$	specific CO_2 impact of energy demand		
	with average reg / nreg contribution	g_{CO2}/kWh	$M_{CO_2} \cdot M^{-1} \cdot L^{-2} \cdot T^2$
P_j^{st}	electric power based on additional energy storage capacities	GW	$M \cdot L^2 \cdot T^{-3}$
H_j	electric losses	GW	$M \cdot L^2 \cdot T^{-3}$
P_{ij}	electric power	GW	$M \cdot L^2 \cdot T^{-3}$
P_j^{reg}	regenerative power	GW	$M \cdot L^2 \cdot T^{-3}$
P_j^{nreg}	non regenerative power	GW	$M \cdot L^2 \cdot T^{-3}$
S_j	total electric power supply	GW	$M \cdot L^2 \cdot T^{-3}$

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7 | LITERATURE

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